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A new class of integrable discrete systems

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Abstract
We derive a new class of integrable autonomous mappings which possess invariants biquartic in the dependent variables. We link these mappings to discrete Painlevé equations. We show how mappings with biquartic invariants can be naturally built up from the requirement that their solutions be parametrized in terms of elliptic functions.

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1. Introduction

The quest for integrable discrete systems has been particularly active over the past decade. Foremost among the results of this collective effort was the discovery of the discrete analogues of the Painlevé equations [1]. To date several methods exist for the systematic derivation of integrable discrete systems. However what remains of capital value are what are colloquially called integrability detectors. They are customarily based on some property which is of universal occurrence in integrable systems to the point that its absence is considered as an indication of nonintegrability. The usual procedure is to postulate a functional form for the discrete system containing enough freedom in the form of adjustable parameters, and fix the value of the latter through the application of the integrability criterion.

One of the first successful integrability criteria proposed was that of the singularity confinement [2]. Its principle is based on the observation that for mappings integrable through spectral methods any spontaneous (i.e. initial condition dependent) singularity disappears after a few iteration steps. The singularity confinement criterion has been particularly useful for the derivation of the discrete Painlevé equations. As was shown recently [3] this criterion is not sufficient because it does not provide control over the growth of the solution of a given mapping at infinity, a feature which, from the results of Ablowitz *et al* [4], is expected to play an important role in integrability. Thus in [5] a combination of this criterion with the

Nevanlinna theory (which is specially tailored to the study of the growth of meromorphic functions) was proposed resulting in a criterion which is expected to be both necessary and sufficient.

A different approach, more directly based on the growth properties of rational mappings, is the one known under the name of algebraic entropy [6]. The latter is a quantity which is greater than zero when the degree of the iterates of some initial data grows exponentially or faster, a property which is considered as an indication of nonintegrability. The method of the study of degree growth has been applied to a host of discrete systems confirming the integrability of previously obtained mappings and helping in the discovery of new ones. It constitutes a well-adapted tool for the exploration of the domain of third-order mappings where very few results exist to date [7].

2. A mapping with biquartic invariant

Before embarking upon the analysis of third-order systems it seemed fit to perform a detailed study of second-order mappings for which several results exist. Quite surprisingly while this domain has been thoroughly studied we obtained novel results which spurred the present study. In a systematic study of mappings using the algebraic entropy method we obtained the following system:

$$\frac{(x_n x_{n+1}-1)(x_n x_{n-1}-1)}{x_{n+1}x_{n-1}} = \frac{\left(x_n^2-1\right)(x_n-a)(x_n-1/a)}{x_n^2-p^2}. \tag{1}$$

In order to obtain the algebraic entropy of a rational mapping we start from $x_0 = 2$ and $x_1 = f$ and count the degree with respect to f for the numerator or denominator of x_n. In the case of (1) we obtained the following sequence of degrees $d_n = 0, 1, 3, 6, 11, 17, 24, 33, 43, 54, 67, \ldots$ which corresponds to quadratic growth. Thus the mapping is expected to be integrable. A direct search for the conserved quantity as a rational function of x_n, x_{n-1} led to the following result:

$$K = \frac{((x_{n-1}-x_n)^2-(p(x_{n-1}x_n-1))^2)\left(\left(x_{n-1}+x_n-\left(a+\frac{1}{a}\right)x_{n-1}x_n\right)^2-(p(x_{n-1}x_n-1))^2\right)}{x_{n-1}^2x_n^2(x_{n-1}x_n-1)^2} \tag{2}$$

i.e. K is a ratio of two biquartic polynomials, i.e. quartic in x_n and x_{n-1} separately. This invariant is quite astonishing. As a matter of fact in all integrable cases known to date the invariant was a ratio of biquadratic polynomials. The systems with an invariant of the latter type belong to the family of QRT [8] mappings. The latter have, in the one-component case, the form

$$x_{n+1} = \frac{f_1(x_n)-x_{n-1}f_2(x_n)}{f_2(x_n)-x_{n-1}f_3(x_n)} \tag{3}$$

where f_i are specific quartic polynomials expressed in terms of 12 parameters of which five correspond to genuine degrees of freedom.

$$K = \frac{\alpha_0 x_{n+1}^2x_n^2+\beta_0x_{n+1}x_n(x_{n+1}+x_n)+\gamma_0\left(x_{n+1}^2+x_n^2\right)+\epsilon_0x_{n+1}x_n+\zeta_0(x_{n+1}+x_n)+\mu_0}{\alpha_1 x_{n+1}^2x_n^2+\beta_1x_{n+1}x_n(x_{n+1}+x_n)+\gamma_1\left(x_{n+1}^2+x_n^2\right)+\epsilon_1x_{n+1}x_n+\zeta_1(x_{n+1}+x_n)+\mu_1}. \tag{4}$$

System (1) is clearly a non-QRT mapping. By this statement we mean that the mapping cannot be found within the QRT parametrization *as it stands*. (This does not preclude the existence of some birational transformation which could reduce the mapping to a QRT form. We shall come back to this point later.)

This is the first ever example of an integrable autonomous second-order system which does not belong to the QRT family. However, its form is tantalizingly close to that of a QRT, in particular if one introduces the change $x \to 1/x$ we obtain

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)\left(x_n^2 - 1\right)}{p^2 x_n^2 - 1}. \tag{5}$$

Again, this mapping, although reminiscent of the mappings of the QRT q-$\mathrm{P_V}$ family,

$$(x_n x_{n+1} - 1)(x_n x_{n-1} - 1) = \frac{(x_n - a)(x_n - 1/a)(x_n - b)(x_n - 1/b)}{(p x_n - 1)(r x_n - 1)} \tag{6}$$

is non-QRT itself: the combination of signs in the parameters of the rhs is just not the right one. When we study the singularities of (5) we find the following patterns: $\{a, 1/a\}$, $\{1/a, a\}$, $\{1, 1\}$, $\{-1, -1\}$, $\{1/p, \infty, 1/p\}$, $\{-1/p, \infty, -1/p\}$. With the exception of the first two, these singularity patterns are atypical. For comparison, the singularity patterns of (6) are $\{a, 1/a\}$, $\{1/a, a\}$, $\{b, 1/b\}$, $\{1/b, b\}$, $\{1/p, \infty, 1/r\}$, $\{1/r, \infty, 1/p\}$. We remark that while for (6) the various roots of the numerator and denominator of the rhs are exchanged in a single singularity pattern for the last four patterns of (5) one enters and exits the singularity through the *same* root.

There exists however a way to explain the existence of a mapping such as (5) and link it to the discrete Painlevé equations. Let us start with the full asymmetric q-$\mathrm{P_V}$ [9]

$$(x_n y_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c)(x_n - d)}{(pq^n x_n - 1)(rq^n x_n - 1)} \tag{7a}$$

$$(x_{n+1} y_n - 1)(x_n y_n - 1) = \frac{(y_n - 1/a)(y_n - 1/b)(y_n - 1/c)(y_n - 1/d)}{(sq^n y_n - 1)(tq^n y_n - 1)} \tag{7b}$$

where the parameters satisfy the constraints $abcd = 1$ and $st = qpr$. Next we try to obtain autonomous reductions of this mapping, but instead of the trivial choice $q = 1$ we take $q = -1$. In order for the mapping to be indeed autonomous we must take $p + r = 0$, $s + t = 0$. Then the constraint becomes $s^2 = -p^2$, or $s = \mathrm{i}p$. Next we introduce the scalings $x \to x\sqrt{\mathrm{i}}$, $y \to y/\sqrt{\mathrm{i}}$, $p \to p/\sqrt{\mathrm{i}}$, $a \to a\sqrt{\mathrm{i}}$, $b \to b\sqrt{\mathrm{i}}$, $c \to c\sqrt{\mathrm{i}}$, $d \to d\sqrt{\mathrm{i}}$ so that we now have $abcd = -1$ instead of 1. We find thus the mapping

$$(x_n y_n - 1)(x_n y_{n-1} - 1) = \frac{(x_n - a)(x_n - b)(x_n - c)(x_n - d)}{p^2 x_n^2 - 1} \tag{8a}$$

$$(x_{n+1} y_n - 1)(x_n y_n - 1) = \frac{(y_n - 1/a)(y_n - 1/b)(y_n - 1/c)(y_n - 1/d)}{p^2 y_n^2 - 1}. \tag{8b}$$

This autonomous mapping is in fact an asymmetric extension of (5). It turns out that it also has a biquartic invariant

$$K = \frac{(x(x - s_1) + y(y - s_{-1}) - (p(xy - 1))^2)^2 - 4(x(x - s_1) + s_2)(y(y - s_{-1}) - s_2)}{(xy - 1)^2} \tag{9}$$

where x, y stand for x_n, y_n or x_n, y_{n-1} and $s_1 = a + b + c + d$, $s_{-1} = 1/a + 1/b + 1/c + 1/d$, $s_2 = (ab + ac + ad + bc + bd + cd)/2$.

From (8) we can obtain the symmetric reduction to precisely (5). We identify $y_{n-1} = X_{2n-1}$, $x_n = X_{2n}$, $y_n = X_{2n+1}$, etc and demand that (8b) be just the upshift of (8a).

The root $1/a$ on the rhs of (8b) should coincide with one of the roots on the rhs of (8a). So unless $a = \pm 1$, without loss of generality one has to assume $1/a = b$ and thus $d = -1/c$. Then, the root $1/c$ on the rhs of (8b) can only coincide with the root c on the rhs of (8a), so $c = -d = \pm 1$. Had we taken $a = \pm 1$ we would have found the same result, up to a renaming of the parameters.

3. Constructing mappings with biquartic invariants

From this construction we see clearly that (5) as well as its asymmetric form (8) are just special, artificially autonomized, cases of (nonautonomous) discrete Painlevé equations. Once this construction has been put forward for q-$\mathrm{P_V}$, it is quite easy to extend it to other families of discrete Painlevé equations and try to obtain autonomous mappings with quartic invariants.

As an illustration we start from the q-$\mathrm{P_V}$ which was introduced in [10]

$$y_n y_{n-1} = \frac{(x_n - aq^n)(x_n - bq^n)}{1 - px_n} \tag{10a}$$

$$x_{n+1} x_n = \frac{(y_n - cq^n)(y_n - dq^n)}{1 - ry_n} \tag{10b}$$

with the constraint $cd = qab$. We try again to obtain an autonomous reduction with $q = -1$. This imposes $a + b = c + d = 0$ and $c^2 = -a^2$. One could leave the mapping under this form, but for reasons that will appear shortly, we introduce a change of variables, $x \to x/\sqrt{pr}$, $y \to \mathrm{i}y/\sqrt{pr}$, $r \to r/\mathrm{i}$. Then the mapping becomes

$$y_n y_{n-1} = \frac{x_n^2 - t^2}{sx_n - 1} \tag{11a}$$

$$x_{n+1} x_n = \frac{y_n^2 - t^2}{y_n/s - 1} \tag{11b}$$

where $s = \sqrt{p/r}$ and $t^2 = pra^2$. The autonomous mapping resulting from this construction is indeed integrable since it is just a subcase of an integrable, nonautonomous discrete Painlevé equation. Moreover, it possesses a biquartic invariant, and again this mapping is *not* of the QRT type:

$$K \equiv \frac{x^2y^2(sy - x/s)^2 + 2xy(x^2 - y^2)(sy - x/s) + 2t^2xy(sy + x/s) + (x^2 + y^2 - t^2)^2}{x^2y^2} \tag{12}$$

where again x, y stand for x_n, y_n or x_n, y_{n-1}.

With this choice of variables, the identification used after equation (9) (but for simplicity, we denote the new variable by x rather than X) leads, in the special case $s = 1/s$, to a symmetric, one-component, form of this mapping

$$x_{n+1} x_{n-1} = \frac{x_n^2 - t^2}{x_n - 1}. \tag{13}$$

The invariant for (13) can be simply obtained from (12):

$$K = \frac{x_n^2 x_{n-1}^2 (x_n - x_{n-1})^2 - 2x_n x_{n-1}(x_n + x_{n-1})((x_n - x_{n-1})^2 - t^2) + \left(x_n^2 + x_{n-1}^2 - t^2\right)^2}{x_n^2 x_{n-1}^2}. \tag{14}$$

It is interesting to study the singularity patterns associated with (13). We find $\{1, \infty, \infty, 1\}$ and $\{\pm t, 0, \pm t\}$. Again, the latter patterns are atypical ones since they do *not* exchange the two roots of the numerator t and $-t$, as the same root appears as the entry and exit point of the singularity.

We have identified one more mapping with biquartic invariant along the same procedure. Contrary to the first two mappings though, it has only an asymmetric form. Starting from q-$\mathrm{P_{VI}}$ [11]

$$y_n y_{n-1} = \frac{(x_n - aq^n)(x_n - bq^n)}{(1 - px_n)(1 - sx_n)} \tag{15a}$$

$$x_{n+1}x_n = \frac{(y_n - cq^n)(y_n - dq^n)}{(1 - ry_n)(1 - ty_n)} \tag{15b}$$

with again $cd = qab$, and in addition $ps = rt$, the choice $q = -1$ leads to an autonomous mapping as before when $a + b = c + d = 0$, with as a consequence $c^2 = -a^2$. We obtain

$$y_n y_{n-1} = \frac{x_n^2 - a^2}{(1 - px_n)(1 - sx_n)} \tag{16a}$$

$$x_{n+1}x_n = \frac{y_n^2 + a^2}{(1 - ry_n)(1 - ty_n)}. \tag{16b}$$

(Here there is no point in trying the change of variables $y \to \mathrm{i}y$ that makes the numerators identical, because the condition would become $ps = -rt$, and then it would be impossible to make the denominators identical.) Calling f the common value of ps and rt, and denoting $g = p + s$, $h = r + t$, the invariant in this case is given by

$$\begin{aligned} x^2y^2K &= x^4y^4f^2 - 2xy(hx + gy)(fx^2y^2 + x^2 + y^2) + x^2y^2(2f(x^2 + y^2) + (hx + gy)^2) \\ &\quad + 2a^2xy(hx - gy) + (y^2 - x^2 + a^2)^2. \end{aligned} \tag{17}$$

4. Interpretation of mappings with biquartic invariant

Although these mappings are by construction integrable one can wonder as to the precise method of their integration. Since all the systems we analysed above are obtained as special limits of q-discrete Painlevé equations (by giving to q a special value) one can in principle use the integration of these Painlevé equations in order to integrate these autonomous mappings as well. The difficulty lies in the fact that not all of the Lax pairs for q-Painlevé equations are known to date. Thus we will suggest another approach, that of the integration of the quartic invariant. As we have shown in [12], the quadratic invariant associated with QRT mappings can be parametrized in terms of elliptic functions. On very general arguments we expect the explicit integration of the mappings presented here to be also in terms of elliptic functions. This would be a further indication as to the existence of a birational transformation reducing these mappings to a QRT form, but as we explain below its explicit construction is of a great practical difficulty. As a matter of fact, the explicit construction of the solution, *even of a given QRT mapping*, is exceedingly difficult. This is due to the fact that the reduction of the biquadratic correspondence associated with the invariant to the canonical form necessitates lengthy (sometimes prohibitively so) calculations. Thus we shall not pursue precisely in this direction. Rather, we shall address the more general question of the possible integration of mappings with biquartic invariants the form of which is inspired by the invariants obtained above.

Our main assumption is that there exist mappings with quartic invariants the solution of which can be parametrized by elliptic functions. Equivalently, we assume that there exist biquartic correspondences which are integrable, and can also be parametrized by elliptic functions.

In the case of the biquadratic correspondence coming from the QRT mapping it was easy to show that $x_n = A\,\mathrm{sn}\,u$, where sn is a Jacobi elliptic function of modulus k, and

$x_{n\pm 1} = A\,\mathrm{sn}(u \pm \phi)$ was, up to a homography, the correct parametrization (with some constraints on A, ϕ). Here we shall generalize this assumption to $x_{n+1} = A\,\mathrm{sn}(u + \phi)$, if n is even, say, and $x_{n+1} = A\,\mathrm{sn}(u + \chi)$ if n is odd, where $\phi \neq \chi$. Using the addition properties of elliptic functions, we can express $x_{n\pm 1}$ in terms of x_n. If n is even,

$$x_{n+1} = \frac{x_n\,\mathrm{cn}\,\phi\,\mathrm{dn}\,\phi + A\,\mathrm{sn}\,\phi\sqrt{\left(1 - x_n^2/A^2\right)\left(1 - k^2x_n^2/A^2\right)}}{1 - k^2x_n^2\,\mathrm{sn}^2\phi/A^2} \tag{18a}$$

$$x_{n-1} = \frac{x_n\,\mathrm{cn}\chi\,\mathrm{dn}\chi - A\,\mathrm{sn}\chi\sqrt{\left(1 - x_n^2/A^2\right)\left(1 - k^2x_n^2/A^2\right)}}{1 - k^2x_n^2\,\mathrm{sn}^2\chi/A^2}. \tag{18b}$$

Since we are looking for a bihomographic mapping in terms of x_{n+1}, x_{n-1}, we eliminate the square roots between the two expressions. This results in the mapping

$$\begin{aligned} x_{n+1}x_{n-1}\left(1 + k^2x_n^2\,\mathrm{sn}\,\phi\,\mathrm{sn}\chi/A^2\right) &+ x_n(x_{n+1} + x_{n-1})\frac{\mathrm{sn}\chi\,\mathrm{cn}\,\phi\,\mathrm{dn}\,\phi - \mathrm{sn}\,\phi\,\mathrm{cn}\chi\,\mathrm{dn}\chi}{\mathrm{sn}\,\phi - \mathrm{sn}\chi} \\ &+ x_n^2 + A^2\,\mathrm{sn}\,\phi\,\mathrm{sn}\chi = 0. \end{aligned} \tag{19}$$

Since this is invariant under the interchange $\phi \leftrightarrow \chi$, (19) is also valid for odd n. We can thus rewrite (19) schematically as

$$x_{n+1}x_{n-1}\left(1 + ax_n^2\right) + cx_n(x_{n+1} + x_{n-1}) + x_n^2 + b = 0. \tag{20}$$

Mapping (20) possesses a biquartic invariant which reads

$$K = \frac{acx^2y^2 + (ab + c^2 - 1)xy + bc}{c(x^2 + y^2) + (c^2 - ab + 1)xy} + \frac{c(x^2 + y^2) + (c^2 - ab + 1)xy}{acx^2y^2 + (ab + c^2 - 1)xy + bc} \tag{21}$$

where x stands for x_n and y for $x_{n\pm 1}$. (Note that K is of the form $L + 1/L$. At each step of the evolution L becomes its inverse, so K is conserved.) On the other hand, (21) can be viewed as a 2–2 correspondence between x and y. Thus the fact that the solutions of the mapping are known in terms of elliptic functions means that (21), as a correspondence, can be parametrized by $x = A\,\mathrm{sn}\,u$ and for y one of the four choices $A\,\mathrm{sn}(u \pm \phi)$, $A\,\mathrm{sn}(u \pm \chi)$. After n iterations, the number of possible images grows only polynomially in n, as the nth iterate must be of the form $A\,\mathrm{sn}(u + j\phi + h\chi)$ with $|j| + |h| \leqslant n$ and $j + h \equiv n \bmod(2)$.

Given the mapping (20), how does one proceed to integrate it for given initial conditions x_0 and x_1? The first step is to compute the *value* of the quantity L (by which we mean the first part of K) from the initial data. Using the explicit form of the solutions one can show that $L = \mathrm{sn}\,\phi/\mathrm{sn}\chi$. Furthermore, the ratio a/b furnishes the ratio k^2/A^4. Substituting back into c we obtain finally for A the biquadratic equation

$$4c^2aA^4 + A^2(K((c+1)^2 - ab)((c-1)^2 + ab) + 2(ab-1)^2 - 2c^4) + 4c^2b = 0. \tag{22}$$

Knowing A we can compute the modulus and using the parameters, obtain $\mathrm{sn}\,\phi$ and $\mathrm{sn}\chi$ since their ratio is already known, and their product is c/A^2. In fact since we have to solve a second-order equation for A^2, we have two choices for this quantity. But this does not lead to different solutions. Calling A_1^2 and A_2^2 the two solutions for the quantity A^2 and similarly the corresponding solutions for k^2, we have $k_1^2k_2^2 = A_1^4A_2^4a^2/b^2 = 1$, and in fact $A_1^2 = k_1^2A_2^2$. When one looks at the consequences of this relation on the elliptic functions this only means that the cn and dn functions are permuted. Since we are working with *a priori* complex quantities, k^2 may well be a complex number and the periods of the elliptic functions need not be real and pure imaginary, but any arbitrary complex numbers. So there is no essential difference between cn and dn, and choosing one or the other solution for A^2 is irrelevant: we have the same functions with different names. If both solutions for A^2 were to be real, it would be more natural to take the smaller one, so that the corresponding choice for k^2 be less than unity, but this is just a matter of convenience. The additional choices of sign, for A itself, and a global sign on $\mathrm{sn}\,\phi$ and $\mathrm{sn}\chi$ (since only their product and ratio are fixed) are also irrelevant: they can be absorbed in a redefinition of the phase u, by either a shift of half a period (sign of A), or a symmetry with respect to the origin (signs of A, ϕ and χ), or both (signs of ϕ and χ). So given (20) and two initial points x_0, x_1 one can reconstruct, up to irrelevant choices of parametrization, all the parameters of the elliptic functions involved in the expression of x_n for arbitrary n.

The degree growth of the iteration of mapping (20) can be easily studied. We find $0, 1, 2, 5, 8, 13, 18, 25, 32, \ldots$ for the initial condition $x_0 = 1$ and $x_1 = f$, which corresponds to a quadratic growth, as expected given the integrable character of this mapping. The singularity pattern of the mapping can be more easily studied if we rescale the dependent variable so as to bring it to the form

$$x_{n+1}x_{n-1}\left(1-x_n^2\right)+\sinh\alpha\,\sinh\beta x_n(x_{n+1}+x_{n-1})+x_n^2-\cosh^2\alpha\,\cosh^2\beta=0. \tag{23}$$

One singularity appears whenever $x_{n-1}\left(1+x_n^2\right) = x_n \sinh\alpha \sinh\beta$ leading to $x_{n+1} = \infty$, with subsequent $x_{n+2} = -x_n$, $x_{n+3} = -x_{n-1}$ satisfying thus $x_{n+3}\left(1+x_{n+2}^2\right) = x_{n+2} \sinh\alpha\,\sinh\beta$, and from there on the singularity is confined. However, four other singularity patterns do exist, $\{\pm\cosh\alpha, \pm\sinh\beta\cosh\alpha/\sinh\alpha, \pm\cosh\alpha\}$ and $\{\pm\cosh\beta, \pm\sinh\alpha\cosh\beta/\sinh\beta, \pm\cosh\beta\}$. They are nonstandard, inasmuch as the singularity is entered and exited through the *same* root. In the light of this result, we can surmise that the presence of such nonstandard singularity patterns may be an indication of the existence of a biquartic invariant for a mapping.

5. Conclusion and outlook

In this paper, we have introduced a new family of second-order, autonomous integrable mappings. After the derivation of the QRT family of integrable mappings more than a decade ago and the subsequent intense activity in the domain of integrable discrete systems, the discovery of this new family was, to say the least, unexpected. Our main result is that second-order autonomous mappings with quartic invariants do exist and we have given several examples of these. What is interesting is that these mappings can be obtained by the autonomization of discrete Painlevé equations. Thus their integrability can be linked to that of the latter, more complicated, systems. The question of the possible integrability of biquartic correspondences, the form of which is inspired by the quartic invariants of the mappings presented here, was also examined. We have shown that if one considers an evolution with two different alternating steps, one can construct a biquartic correspondence which is parametrized exactly by elliptic functions.

Several questions remain open at this stage. The first concerns the precise integration of the mappings we obtained. Although we expect the general integrable second-order mapping with biquartic invariant to be integrable in terms of elliptic functions, we surmise that the explicit construction of the solution may turn out to be exceedingly cumbersome in most cases. Given this difficulty, the relation of these mappings with discrete Painlevé equations is an indication that their integration may also be carried out in a different, perhaps simpler, way. The discovery of one class of second-order integrable mappings which do not belong to the QRT family raises legitimately the question of the existence of other such classes. This is reinforced by the discovery of another family of integrable second-order mappings which go beyond the QRT parametrization, recently derived by Roberts and Iatrou [13].

We expect the dual approach, integrability detection (based on degree growth and singularity analysis techniques) and construction of solutions (assuming a given family of functions) to be particularly useful in this direction.

References

[1] Grammaticos B, Nijhoff F and Ramani A 1999 Discrete Painlevé equations *The Painlevé Property: One Century Later (CRM Series in Mathematical Physics)* ed R Conte (New York: Springer) pp 413–515

[2] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825

[3] Hietarinta J and Viallet C 1998 *Phys. Rev. Lett.* **81** 325

[4] Ablowitz M J, Halburd R and Herbst B 2000 *Nonlinearity* **13** 889

[5] Grammaticos B, Tamizhmani T, Ramani A and Tamizhmani K M 2001 *J. Phys A: Math. Gen.* **34** 3811

[6] Bellon M P and Viallet C-M 1999 *Commun. Math. Phys.* **204** 425

[7] Hirota R, Kimura K and Yahagi H 2001 *J. Phys. A: Math. Gen.* **34** 10377

[8] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* D **34** 183

[9] Ramani A, Grammaticos B and Ohta Y 1996 *CRM Proceedings and Lecture Notes* vol 9, ed D Levi, L Vinet and P Winternitz (Providence, RI: American Mathematical Society) p 303

[10] Ramani A, Grammaticos B and Ohta Y 2001 *J. Phys. A: Math. Gen.* **34** 2505

[11] Jimbo M and Sakai H 1996 *Lett. Math. Phys.* **38** 145

[12] Ramani A, Carstea A S, Grammaticos B and Ohta Y 2002 *Physica* A **305** 437

[13] Iatrou A and Roberts J A G 2001 *J. Phys. A: Math. Gen.* **34** 6617